COMMENTARY
SOLVING WORD PROBLEMS: A CASE OF MODELLING?

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Abstract
According to the researchers who report in this set of papers there are two causes for the phenomenon that primary- and secondary-school students ignore relevant and plausibly familiar aspects of reality in answering word problems. The first cause is the stereotyped character of common word problems. The second cause is the classroom climate. In this article, it is argued that the students act sensibly in their situation. Furthermore, it is noted that the use of stereotyped problems and the accompanying classroom climate relate to teacher beliefs about the goals of mathematics education. Therefore, improving the results on problematic word problems will ask for a change in teacher beliefs. Furthermore, a directed effort to change the classroom socio-math norms will be needed. In relation to this, Greer's suggestion for a change towards a modelling perspective is supported. However, what modelling is, is worked out differently. In line with the RME instructional theory, a plea is made for modelling as an activity of organizing, not of translation. © 1997 Elsevier Science Ltd

Introduction
The research reported in this special issue shows that primary- and secondary-school students ignore relevant and plausibly familiar aspects of reality in answering word problems. The authors point to two causes for this abstention from using everyday-life knowledge. The first cause is in the stereotyped character of common word problems. The second cause is in the classroom climate. Stereotyped problems do not ask for reflection. In fact, the character of the problems is such that the students have to refrain from realistic considerations—and, in general, the classroom climate is one that endorses this separation between school mathematics and everyday-life reality. The students are not expected to reflect upon their answers. The focus is on getting right answers quickly, and not on reasoning about the represented real-life situation.

In the following, I will start with an analysis of the aforementioned causes, as a lead-in to a discussion of the possible remedies. This discussion will especially address the "modelling perspective" that is proposed by Greer (1997).
Stereotyped Problems

Most text-book word problems are nothing more than poorly disguised exercises in one of the four basic operations. In general, these problems seldom ask for more than one operation. So, for the students, the name of the game becomes finding the proper operation and executing it. Given the stereotyped character of common word problems, the behaviour of the students can be judged to be quite sensible.

Their behaviour shows commonalities with the dairy workers or the candy sellers in the well-known research on situated cognition. Like the candy sellers and the factory workers, they adapt their behaviour to the tasks and their setting. They use the characteristics of their situation to develop behavior that is efficient in that situation.

This interpretation also resonates with a more general theory of routine behaviour. According to van Parreren (1972), the development of human behaviour can be characterized by a strive for efficiency. Generalization and the development of routine behaviour are typical manifestations thereof. In practice, routine behaviour develops even further, to become automated actions which are no longer consciously executed. Van Parreren calls this "valentie handelen" (van Parreren, 1972). Let me give an example.

When you have to learn to find your way in an unfamiliar city, finding your way at first manifests itself as a problem to be solved. After some time, after having travelled the same road a number of times, finding your way becomes a routine activity. Finally, you know this route so well that you do not even have to think where you are going: the phase of valentie handelen has been entered. This lack of consciousness may show in several ways. For instance, when the route from A to B partly coincides with the route from C to D, it may happen that you find yourself heading for B, although you intended to go to D. Another instance is when you meet a blockade and you cannot proceed on your normal route. Then you find that you have to step back, to consider where you are to be able to figure out an alternative route. Similarly, when someone ask you to tell her how to get from A to B, you have to retrace the whole journey in your mind to be able to give directions. Even when you try to do this faithfully, you may skip some crucial detail unknowingly.

In a similar manner, primary school students may not be aware of the strategies they use when solving word problems. This may be the reason why general warnings, along the lines of "these problems may need more careful consideration", do not seem to have much effect (Reusser & Stebler, 1997; Yoshida, Verschaffel, & De Corte, 1997).

I would like to stress that I do not see valentie handelen as such as something objectionable. On the contrary, automated actions enlarge our possibilities. Moreover, this type of behaviour is effective in 99 out of 100 cases. (If it was not, we would not maintain it.) Automated behavior is not senseless, it can be very sensible in a certain situation. In the case of stereotyped word problems, the issue is not so much with automated behavior as such, but with the fact that the behaviour that is developed is only useful in the context of a very limited form of school mathematics.

The Function of Word Problems

In light of the above, the most logical reform recommendation seems to be to introduce more variety into word problems. Word problems should be enriched by: "including
superfluous data, or omitting data within a problem; requiring estimation rather than exact computation; using complex, multistep problems; requiring students to formulate problems themselves" (Greer, 1997). However, these recommendations are not new (see, for instance, van Gelder, 1969). Moreover, it does not seem to be a major task to create such problems. Still, in many countries the textbooks have not changed, and we may wonder:

(1) Why don't textbook authors include this variety more systematically in their products?
(2) Maybe because teachers do not ask for it?
(3) Or, maybe because teachers would even resist such changes?

The latter hypothesis seems to be supported by the research of Verschaffel, De Corte, and Borghart (1997) among student teachers in Belgium. In this research, student teachers were asked to judge realistic and non-realistic answers to problematic word problems. These are word problems "in which the underlying mathematical modelling assumptions are problematic from a realistic point of view" (Verschaffel et al., 1997). The research showed that the student-teachers' overall evaluation of the stereotyped, non-realistic answers to these problematic items was considerably more positive than for the realistic answers based on context-based considerations. Apparently, these student teachers differ from the researchers in their beliefs about the importance of realistic considerations in relation to word problems. The teachers seem to believe that the activation of realistic context-based considerations should not be stimulated but rather discouraged in elementary school mathematics.

This indicates a difference in views on the function of word problems in mathematics education. The researchers relate word problems to problem solving and applications. The student teachers (and teachers in general, probably) see another role for word problems. It seems likely that for them the primary role of word problems is that of exercises. In such a conception, word problems are nothing more, and nothing less (!) than decorated exercises in the four basic operations. This is in line with Wyndhamm and Säljö's assertion that word problems are recontextualized forms of decontextualized descriptions of everyday-life situations, that serve a specific purpose:

The construction of a word problem describing an everyday set of events implies that an account of a situation (formulated in everyday language) is decontextualized and subsequently recontextualized in a different setting (as a word problem in mathematics teaching) (Wyndhamm & Säljö, 1997).

In this process of decontextualization and recontextualization, they argue, the premises for communication are changed. The problem description changes into something that is an instance of a particular kind of exercise.

If training on this kind of exercises is our objective, it makes sense, not to add variation to word problems. That would only complicate matters, take up extra time and effort, and would undermine the intended practice. I do not want to imply that we therefore should give up on "sense-making". On the contrary. However, I do think that we have to make an explicit choice between computational proficiency and sense-making. I would argue in favour of sense-making, certainly in light of the availability of calculators and other technology in today's society. This would mean a reorientation concerning the purpose and goals of modern mathematics education.

This is, in essence, what the authors of this set of papers argue for too when they plea for a change in classroom culture towards a modelling perspective.
Broadly speaking, the view is that it is not a cognitive deficit as such that causes the absten-
tion from sense-making, but rather that the children are acting in accordance with a typical
school mathematics classroom culture. Or as Reusser and Stebler (1997) put it:

Doing mathematics, including word problems, or knowing what mathematics in the structured social context
of schooling is all about, is inseparable from the (micro)cultural web of socio-cognitive practices of instruc-
tion, and of the materials employed.

To change this, one has to change the didactical contract (Brousseau, 1990). The notion
“didactical contract” refers to the set of reciprocal expectations and obligations between the
teacher and the students that has evolved in their ongoing interaction. A renegotiation of
the didactical contract should make it “more reasonable for students to pay attention to the
referential meaning of statements... which encourage such a line of thinking while solving
word problems” (Wyndham & Säljö, 1997).

In discussions on classroom culture in relation to reform in mathematics education, the emphasis is often on those classroom norms that can be characterized by the adage, “explain
and justify”. The students are expected to explain and justify their own ideas and solutions,
and they are expected to try to understand the ideas and solutions of others, ask for clarifica-
tion, and challenge them if necessary. In the case of word problems, however, we are not as
much dealing with general social norms, but with norms that more specifically concern the
mathematics. Yackel and Cobb (1995) call these socio-math norms. These socio-math norms
deal with such issues as: what counts as a problem, and what counts as a solution. Whether
to take realistic considerations into account or not, typically concerns these issues. Albeit, it
is not just a matter of “yes, we do”, or “no, we don’t”. For there is an insurmountable dif-
fERENCE BETWEEN SOLVING AUTHENTIC PROBLEMS IN REALITY AND SOLVING WORD PROBLEMS IN SCHOOL MATH (“T)HE REALISM OF THE REALISTIC WORD PROBLEMS IS NOT IDENTICAL TO THE REALISM WE
WOULD PERCEIVE IF ACTING IN A DIFFERENT CONTEXT” (WYNDHAM & SÄLJÖ, 1997). In the classroom,
one is always dealing with a reduction of reality. The question for the students, however, is
what level of reduction is expected.

Let me illustrate this with the example in Fig. 1.

Possible solutions in the real situation might be:

- Marco shares his cheeseburger with his friend.
- Father and mother share one cheeseburger to help out.
- Someone goes to the shop to buy an extra one.

**Problem**

Marco asks his mother if his friend Pim may stay for dinner. His mother
agrees, but this means that there is one cheeseburger short. There are
five cheeseburgers, and including Pim there are six people now.

![Cheeseburgers](image)

How would you divide five cheeseburgers between six people?

Figure 1. A realistic problem.
How do the students "know" that this type of answer is not expected from them, while in other instances realistic considerations are called for? According to Yackel and Cobb (1995), such socio-math norms are interactively constituted in a process of negotiation.

The classroom norms are reflexively related with the individual beliefs of the students. On the one hand, the norms are constituted by the totality of the individual beliefs of the students, and, on the other hand, the individual beliefs are shaped by the norms which are experienced in the classroom. It is in the interaction in the classroom, where the students offer solutions, and the teacher reacts to them, that the social norms are constituted. That is how the students learn in how far to take realistic aspects into account.

In traditional classrooms the norm is: don't bother with reality, just focus on the mathematics. As far as students are expected to explain or justify their solution, the discussion is about the calculational aspect and not about the quantitative meaning (Thompson, 1993). To change this socio-math norm, both the word problems and the manner in which they are discussed have to change. To change the socio-math norms, the teacher has to use concrete instances to make the intended norms a topic of discussion (Cobb, 1991). Furthermore, the teacher has to consistently show that he or she values realistic considerations over flawless but senseless calculations. Only if the teacher is consistent in what he or she values, may students be expected to change their beliefs on what the norms are.

A special case of realistic considerations consists of ambiguities like counting or discounting the serrated edge in the case of sawing planks of one metre long from planks of two metres long (Greer, 1997). Here, not only socio-math norms, but also "real" realistic considerations have to be looked at. In the actual activity of sawing planks, one will in general disregard the inaccuracy due to the serrated edge. Only in very special cases would such a high precision be asked for. In my view, this is an important aspect of real-world problems. The students have to learn to deliberate on estimation and rounding in relation to what precision the portrayed situation asks for.

Modelling

As a spokesman of the other authors, Greer calls for a modelling perspective.

The main trust of our recommendations is to treat word problems as exercises in modelling (or mathematization). Modelling may be viewed as the link between the "two faces" of mathematics, namely its grounding in aspects of reality, and the development of abstract formal structures (Greer, 1997).

Modelling, however, can be conceived in two different ways:

1. as a form of translation, or
2. as a form of organizing.

Polya, for instance, states that: "the student translates a real situation into mathematical terms" (Polya cited in Reusser & Stebler, 1997), while Freudenthal (1971) speaks of "organizing a subject matter". The translation interpretation construes the reality and the model as two disjunct entities. Greer's recommendation seems to fit this category. In his view, addition, subtraction, multiplication, and division are conceptualized as providing potential models for situations. Students have to become aware of the distinction between the model and the situation, and they have to learn to assess whether the model is more or less adequate against the backdrop of contextual factors such as the goals of the modeller.
This model-situation duality is worked out in a schematization of the modelling process (see Fig. 2).

I want to contrast this with the organizing approach that emphasizes modeling as an activity. In this conceptualization, a model is the result of an organizing activity. It is in the process of structuring a problem situation that the model emerges. Consequently, such models may be less sophisticated and more informal than the mathematical models Greer has in mind. The organizing activity may give rise to informal ways of modeling that may show traces of the situation modeled.

This can be illustrated by the solution of a third grader of the division problem presented in Fig. 3 (van Galen, Gravemeijer, Kraemer, Meeuwisse, & Vermeulen, 1985).

The student invented the solution procedure shown in Fig. 4.

The sweets are distributed one by one. One by one, they are crossed out of the total and added to one of the rows. The informal character of the model is underlined by the fact that the student even tried to copy the children's portraits faithfully.

The idea of modeling as organizing fits with Freudenthal's notion of mathematics as a human activity. For Freudenthal (1971, 1973, 1991), mathematical activity mainly consists of organizing, or mathematizing. Over the years, Freudenthal's ideas have been worked out in the domain-specific instruction theory for "realistic mathematics education" (RME) (Gravemeijer, 1994; Treffers, 1987). In RME, modeling as an activity is further elaborated in a didactical sense. The idea is that informal ways of modelling emerge when the students are organizing contextual problems. Later, these ways of modelling serve as a basis for developing formal mathematical knowledge. To be more precise, at first a model is constituted as a context-specific model of acting in a situation, then the model is generalized over situations. Thus, the model changes character, it becomes an entity of its own, and in this new shape it can function as a model for more formal mathematical reasoning.

We may take long division as an example. Here solving a division problem in reality can be modelled by repeated subtraction. Repeated subtraction can truly function as a model of real-life division, for concrete strategies like handing out more items at the time can be modelled by a curtailment of the subtraction. A curtailed repeated subtraction scheme, in turn, can function as a model for reasoning about the standard algorithm.

The distinction between a "model of" and a "model for" can be characterized as a distinction between a "referential level" and a "general level" (Gravemeijer, 1994). At the referential level, the model refers to the situation sketched in the problem statement. The model is meaningful to the student, because of this reference to a concrete situation. When the student gains more experience with acting with this model, the attention shifts from the original

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Figure 2. The core of a modelling process, according to Greer.
Three children divide 36 sweets. How many sweets will each get?

Figure 3. A problem of dividing sweets.

Figure 4. Piecewise division of a collection of sweets.

situation to the mathematical relations involved. The process of acting with the model is gradually reified, and, at the general level, the student no longer needs to think of the problem situation to give meaning to the model. This "reified model" then can function as a model for mathematical reasoning.

In total, we may discern four levels: a situational level, a referential level, a general level, and a formal level (Gravemeijer, 1994). The first level concerns acting in the real situation,
where domain-specific, situational knowledge and strategies are used to solve concrete problems. These actions are modelled on the referential level. Then the shift towards the general level is made. And finally, one may envision a forth level, a formal level, where the students reason in terms of formal mathematical relations.

Note that the model does not have to be static. The student's solution of $36 \div 3$ by piecewise distribution, with the help of hash marks, can be seen as an informal pre-model. In the next step numbers are introduced, and the division is modelled by repeated subtraction. At first, this repeated subtraction may be rather straightforward. Later, curtailment will change its appearance, and it may finally develop into a rather sophisticated procedure.

The rationale for this modelling approach in RME is twofold. Firstly, the model-of/model-for progression has a didactical function. The main idea is that the progressive mathematization leads to algorithms, concepts, and notations that are rooted in a learning history that starts with informal experientially real knowledge of the students. The roots in the student's reality are expected to foster the meaningfulness and usefulness of the so-developed mathematical knowledge. Secondly, when modelling is constituted as an organizing activity, the students practice this form of mathematizing. This experience in mathematizing then will be beneficial when students have to tackle applied problems. In this view, students are expected to approach an unfamiliar problem as a situation to be mathematized, not primarily as a situation for application of ready-made solution procedures. That does not mean that the student's knowledge of solution procedures does not play a role, but the primary objective of the student would be to make sense of the problem. In practice, it will often be a matter of shuttling back and forth between the interpretation of the problem and a review of possible suitable procedures. However, if the student does not have fitting ready-made solution procedures available, an organizing/modelling approach may enable the student to find an adequate informal solution. Moreover, the ability to organize, and the attitude to approach a new problem as a situation to be mathematized, may have a more general value than any ready-made solution procedure.

Concluding Remarks

Recent research on word problems has revealed the complex nature of the processes that lead to the lack of activation of real-world knowledge. This is due to the fact that the word-problem research community did not limit itself to the study of task characteristics in isolation, but widened its scope and included social interaction and the setting of word-problem arithmetic in schools. In line with this development, Greer proposes a shift towards a modelling perspective (Greer, 1997). However, my recommendation would be to go one step further, and to focus on modelling as a form of organizing, instead of an act of translation. I want to repeat that this implies a major reorientation on the goals of mathematics education. Such a reorientation can be cast in the context of a fundamental consideration of the goals of mathematics education for the next century.

The traditional mathematics curriculum for primary school has its roots in the past, in times when smooth and flawless execution of written algorithms was an ability with high societal and economic value. But that is something of the past. Calculators and computers are taking over all the laborious arithmetical work. At the same time, the modern citizen is
A CASE OF MODELLING?

bombardeed with numerical and statistical data. One has to be able to deal with this information on a different level—being able to judge the meaningfulness and the correctness of calculations, being able to judge the plausibility or the true meaning of data. Those will have to be the goals for the future. We have to acknowledge, however, that setting an agenda is not the same as implementing it in practice. Therefore, educational psychology research should continue to investigate in how far good intentions are accompanied by the intended results.

Acknowledgments—The preparation of this paper was supported by the National Science Foundation under grant number RED-9353587. All opinions expressed are solely those of the author.

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