Local Instruction Theories as Means of Support for Teachers in Reform Mathematics Education

Koeno Gravemeijer
Freudenthal Institute & Department of Educational Research
Utrecht University

This article focuses on a form of instructional design that is deemed fitting for reform mathematics education. Reform mathematics education requires instruction that helps students in developing their current ways of reasoning into more sophisticated ways of mathematical reasoning. This implies that there has to be ample room for teachers to adjust their instruction to the students' thinking. But, the point of departure is that if justice is to be done to the input of the students and their ideas built on, a well-founded plan is needed. Design research on an instructional sequence on addition and subtraction up to 100 is taken as an instance to elucidate how the theory for realistic mathematics education (RME) can be used to develop a local instruction theory that can function as such a plan. Instead of offering an instructional sequence that 'works,' the objective of design research is to offer teachers an empirically grounded theory on how a certain set of instructional activities can work. The example of addition and subtraction up to 100 is used to clarify how a local instruction theory informs teachers about learning goals, instructional activities, student thinking and learning, and the role of tools and imagery.

INTRODUCTION

In the 1960s and 1970s theories for instructional design were in vogue in the educational-research community. The most well-known design theories from that period are probably Gagné's Principles of Instructional Design (Gagné & Briggs, 1974). Since then, the interest for instructional design has faded away. More re-
ently, however, a renewed interest can be noticed, especially in communities of mathematics educators. This relates to the current reform efforts in mathematics education. Constructivism formed one of the catalysts of this reform movement. Various interpretations of constructivism fueled the belief that mathematics education should capitalize on the inventions by the students. This in part can explain why there initially was little interest in instructional design within this reform movement. We can even put it in stronger terms: For many, instructional design was seen as incompatible with mathematics education that put the students’ own ideas and input at the forefront. This gradually changed, and the insight is growing that mathematics education that aims to capitalize on the input of the students requires thorough planning. However, this would mean a different kind of planning than envisioned in traditional instructional design strategies.

The instructional design principles of the 1960s and 1970s do not fit reform mathematics instruction. The main problem is that the older design principles take as their point of departure the sophisticated knowledge and strategies of experts to construe learning hierarchies. Following a task analysis approach, the performance of the expert is taken apart and laid out in small steps, and a learning hierarchy is constituted that describes what steps are prerequisite and in what order these steps should be acquired. The result is a series of learning objectives that can make sense from the perspective of the expert, but not necessarily from the perspective of the learner. Further, there is little room for personal input from the learner.

What is needed for reform mathematics education is a form of instructional design supporting instruction that helps students to develop their current ways of reasoning into more sophisticated ways of mathematical reasoning. For the instructional designer this implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing.

This change of perspective encompasses both a change in pedagogy and a change in curriculum; reform mathematics asks for a specific classroom culture and discourse but it also asks for another curriculum and corresponding instructional activities. In the current discussion on reform in mathematics education, the former often get the most attention. The reform pedagogy is elaborated in terms of classroom culture, social norms, mathematical discourse, mathematical community, and a stress on inquiry and problematizing. Without denying the importance of this aspect of reform, it could be necessary to draw the attention to the curriculum counterpart of this innovative pedagogy.

The central problem in reform in mathematics teaching is the well-known tension between the openness toward the students’ own constructions and the obligation to work toward certain given endpoints. Or as Deborah Ball (1993) noted:

How do I create experiences for my students that connect with what they know and care about but also transcend the present? How do I value their interests and also con-
nect them to ideas and traditions growing out of centuries of mathematical exploration and invention? (p. 375)

It is a question of how the teacher could proactively support students' mathematical development, as Cobb (1996) noted. The pedagogy mentioned ensures openness toward students' own constructions. Working toward given end goals asks for more. It asks for what Simon (1995) called "hypothetical learning trajectories." The teacher has to envision how the thinking and learning in which the students could engage as they participate in certain instructional activities relates to the chosen learning goal. Simon emphasized the hypothetical character of these learning trajectories; the teachers must analyze the reactions of the students in light of the stipulated learning trajectory to find out to what extent the actual learning trajectory corresponds with what was envisioned. Based on this information the teacher has to construe new or adapted instructional activities in connection with a revised learning trajectory.

The example Simon (1995) worked out shows that designing hypothetical learning trajectories for reform mathematics is no easy task. We can, therefore, ask ourselves what kind of support can be given to teachers. It is clear that we cannot rely on fixed, ready-made, instructional sequences, because the teacher will continuously have to adapt to the actual thinking and learning of his or her students. Thus, it seems more adequate to offer the teacher some framework of reference, and a set of exemplary instructional activities that can be used as a source of inspiration.

This is exactly what is aimed for in the Dutch tradition of developing realistic mathematics education (RME). Here the objective is to design support materials by trying to construe learning paths along which students could reinvent conventional mathematics. Such a learning path is paved with instructional activities that can function as stepping stones in this conjectured reinvention process. The conceptualization of these learning paths is of the same character as that of Simon's (1995) learning trajectories. Significant differences with hypothetical learning trajectories, however, are the duration of the learning process, and the "situatedness" in a specific classroom, or more to the point, the lack thereof. To emphasize the distinction, I reserve the term hypothetical learning trajectories for the planning of instructional activities in a given classroom on a day-to-day basis, and I use the term local instruction theories to refer to the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic (e.g., addition and subtraction up to 20, area, fractions, etc.).

The relation between hypothetical learning trajectories and local instruction theories can be elucidated with Simon's (1995) travel metaphor. In terms of a travel metaphor, the local instruction theory offers a "travel plan," which the teacher has to transpose into an actual "journey" with his or her students. The idea is that the teachers use their insight in the local instruction theory to choose instructional activities and to design hypothetical learning trajectories for their own students. In
my view, local instruction theories can never free the teachers from having to design hypothetical learning trajectories for their own classroom. Nevertheless, I would argue, that using a local instruction theory as a framework of reference could enhance the quality of the learning trajectories.

This, in a sense, is the main point of this article: Externally developed local instruction theories are indispensable for reform mathematics education. It is unfair to expect teachers to invent hypothetical learning trajectories without any means of support. In addition, it can be argued that without them, the chances to reconcile openness toward students’ own contributions and aiming for given end goals are very slim.

To develop local instruction theories to support teachers, a theory is needed on how to help students’ construct mathematical ideas and procedures. The point of departure here is that RME offers such a theory and that design research is the appropriate method for developing local instruction theories. I elucidate this in the following way. I start with a description of design research as a method for developing local instruction theories. Next I use this as a background to elucidate the RME theoretical framework on the basis of an exemplary local instruction theory. I complement this by highlighting the very aspects in which the local instruction theory goes beyond the level of an instructional sequence in terms of a series of instructional activities.

**DESIGN RESEARCH**

Local instruction theories are developed in what is called developmental research (Gravemeijer, 1994, 1998) or design research (Gravemeijer & Cobb, 2001). The core of this type of research is formed by classroom teaching experiments that center on the development of instructional sequences and the local instructional theories that underpin them. In the course of a classroom teaching experiment, the research team develops sequences of instructional activities that embody conjectures about the course of students’ learning. To this end, the designer conducts an anticipatory thought experiment by envisioning both how proposed instructional activities can be realized in interaction in the classroom and what mental activities students can engage in as they participate in them. Analyses of the actual process of students’ mental activities when they participate in the instructional activities as constituted in the classroom can then provide valuable information that can be used to guide the revision of the instructional activities. The rationale for the instructional sequence can be conceived as a local instructional theory that underpins a prototypical instructional sequence (Gravemeijer, 1994, 1998).

The design research at the Freudenthal Institute grew out of the desire to develop mathematics education that corresponds with Freudenthal’s (1973, 1991) ideal of “mathematics as an human activity.” According to Freudenthal, students...
should be given the opportunity to reinvent mathematics by mathematizing—
mathematizing subject matter from reality and mathematizing mathematical sub-
ject matter. In both cases, the subject matter that is to be mathematized should be
experimentally real for the students. That is why this approach is named RME. One
of the core principles of RME is that mathematics can and should be learned on
one’s own authority and through one’s own mental activities. That is to say, stu-
dents should experience the process by which new mathematics is learned as a
reinvention process in which they themselves play an active role. Moreover, they
should develop sufficient intellectual autonomy (Kamii, Lewis, & Livingstone
Jones, 1993) to only accept new mathematical knowledge of which they can judge
the validity themselves. Within the RME research community, the question is
asked what mathematics education would have to look like to fulfill the previously
discussed educational philosophy by experimenting with mathematics education
in practice and by reflecting on this experimental practice. This reflection leads to
the development of an educational theory, and this theory feeds back into new ex-
periments. This implies that the resulting theory that I call “a domain-specific in-
struction theory for realistic mathematics education” (or RME theory, for short) is
always under construction.

In conjunction with the theory, a research method emerged in the Netherlands
that was labeled developmental research.¹ Similar approaches emerged elsewhere,
for instance, under the names of design experiments (e.g. Brown, 1992; Cobb,
McClain, & Gravemeijer, 2003) and design research (e.g. Edelson, 2002).² In this
article, I use the latter term, design research, which seems to be more common than
the Dutch label of developmental research.

Preliminary Design

Design research that focuses on the development of local instruction theories basi-
cally encompasses three phases: developing a preliminary design, conducting a
teaching experiment, and carrying out a retrospective analysis. The first phase
starts with the clarification of mathematical learning goals, combined with antici-
patory thought experiments in which one envisions how the teaching-learning pro-
cess can be realized in the classroom. This first step results in the explicit formula-
tion of a conjectured local instruction theory that is made up of three components:
(a) learning goals for students, (b) planned instructional activities and the tools that

¹Accidentally, the same label, developmental research, is used for another Dutch research approach.
Instructional design is also at the heart of this research approach; however, the goal is not to develop do-
main specific instruction theories but to develop improved design theories (van den Akker, 1999).
²It should be noted that the description of developmental research presented here also builds on
what is learned in the collaboration of the author with Paul Cobb and Kay McClain at Vanderbilt Uni-
versity (see also Gravemeijer & Cobb, 2001).
will be used, and (c) a conjectured learning process in which one anticipates how students’ thinking and understanding could evolve when the instructional activities are used in the classroom.

This conjectured local instruction theory is open to adaptations on the basis of input of the students and assessments of their actual understandings. The theory also reflects the importance of anticipating the possible process of their learning as it could occur when planned instructional activities are used in the classroom. The manner in which a conjectured local instruction theory is construed can be described as “theory-guided bricolage” (Gravemeijer, 1994) because it resembles the manner of working of an experienced “tinkerer,” or “bricoleur.” The design researcher follows a similar approach using and adapting existing ideas and materials, but the way in which selections and adaptations are made is guided by a theory—in our case, RME theory. RME theory offers three design heuristics, denoted as guided reinvention, didactical phenomenology, and emergent modeling (which I discuss in more detail later). These design heuristics help the research team in designing a possible learning route together with a set of potentially useful instructional activities that fit this learning route. More specifically, this implies that the researchers think through what mental activities of the students can be expected when they engage in the instructional activities and how those mental activities can help the students to develop the envisioned mathematical insights. In the teaching experiment, those conjectures are put to the test.

The Teaching Experiment

This process of anticipating and testing is, in fact, an iterative process that resembles Simon’s (1995) “mathematics teaching cycle.” The actual enactment of the instructional activities in the classroom enables the researchers to investigate whether the mental activities of the students correspond with the ones they anticipated. The insights gained in this manner and the experience with the instructional activities as such form the basis for the design or modification of subsequent instructional activities and for new conjectures about what mental activities of the students can be expected. In this manner, instructional activities are tried, revised, and designed on a daily basis during the teaching experiment. This cyclic process of thought experiments and instruction experiments (Freudenthal, 1991) forms the backbone of the design research method employed in the teaching experiment (see Fig. 1).

Even though the researchers carry out thought experiments and instruction experiments on a daily basis, the goal of the research team is not to prepare the next day’s instructional activity, but to develop a well-considered and empirically grounded local instruction theory. The term local instruction theory is coined to convey the intention of offering more than a description of a learning route, or the corresponding instructional activities. In addition to these two, a local instruction theory also includes a rationale. In contrast with traditional instructional design re-
search, the objective of design research is not to offer an instructional sequence that "works," but to offer the user an empirically grounded theory on how the researchers think that a certain set of instructional activities could work (cf. NCTM Research Advisory Committee, 1996).

This is in line with the earlier described notion of a travel plan; it does not seem plausible that an instructional sequence could be enacted in the exact same manner in a variety of classrooms—especially if we expect the teacher to adapt to the students' thinking. In contrast, individual teachers can use a local instructional theory as a framework of reference for the design of hypothetical learning trajectories that fit the actual needs of their students.

In the design research project, the mathematical teaching cycles serve the development of the local instruction theory. In fact, there is a reflexive relation between the thought and instruction experiments and the local instruction theory that is being developed. On one hand, the conjectured local instruction theory guides the thought and instruction experiments, and on the other hand, the micro instruction experiments shape the (conjectured) local instruction theory (Fig. 2).

To be able to adjust the envisioned instructional activities on a daily basis, it is desirable that the researchers be present in the classroom every day while the teaching experiment is in progress. The ongoing analyses of individual children's activity and of classroom social processes inform new anticipatory thought experiments in the course of which conjectures about possible learning trajectories are frequently revised. As a consequence, there is often an almost daily modification of local learning goals and instructional activities.

This focus on the ongoing process of experimentation emphasizes that ideas and conjectures are modified while interpreting students' reasoning and learning in the classroom. The empirical data on the activities of the students are interpreted in light of the theoretical framework of RME. In addition to this, the interpretive
framework of Cobb and Yackel (1996) can be used to help the researchers make sense of classroom events.

Retrospective Analysis

The results of design experiments cannot be linked to pretest and posttest results in the same direct manner as is common in standard formative evaluation, because the proposed local instruction theory and prototypical instructional sequence will differ from what is tried in the classroom. Because of the cumulative interaction between the design of the instructional activities and the assembled empirical data, the intertwinemement between the two has to be unraveled to pull out the optimal instructional sequence in the end. For it does not make sense to include activities that did not match their expectations, but the fact that these activities were in the sequence will have affected the students. Therefore, adaptations will have to be made when the nonfunctional, or less functional, activities are left out.

Consequently, the instructional sequence is put together as a reconstruction of a set of instructional activities, which are thought to constitute the effective elements of the sequence. This reconstruction of the optimal sequence is based on the deliberations and the observations of the research team. In this manner, the result of a developmental research experiment is a well-considered and empirically grounded rationale for the envisioned learning route in connection with the proposed set of instructional activities. Methodologically, this result has to be justified by the learning process of the research team. In relation to this, we can refer to a method-

3 The retrospective analysis can spark ideas that surpass what is tried out in the classroom. This can create the need for a new developmental research project, starting with a new conjectured local instruction theory. In this manner, subsequent teaching experiments can become part of a series of macrocycles of experimentation and revision.
olgical norm of trackability (Smaling, 1987, 1992), which is common in ethnography; outsiders should be able to retrace the learning process of the research team. Insight into this learning process, which is shaped by the observations and the deliberations of the research team, should enable outsiders to assess the viability of the results. This again relates to the goal of producing viable theories that can be adapted by classroom teachers. Moreover, understanding the how and why enables the teachers to extend the design research to their own practice, within which they experiment with the conjectured local instruction theory. Actually, feedback from teachers can inform the researchers about the different ways in which the theory can be adapted to various classroom situations.

EXEMPLARY LOCAL INSTRUCTION THEORY

In the following, I discuss the core elements of a local instruction theory on the basis of an instructional sequence that is developed in a teaching experiment in Nashville, Tennessee by Cobb, Gravemeijer, McClain, and Stephan of Vanderbilt University (Stephan, Bowers, Cobb, & Gravemeijer, 2000). But, before focusing on the design of the instructional sequence and the corresponding local instruction theory, I want to stress the importance of the classroom culture that is essential for the enactment of such an instructional sequence. To realize a problem-centered, or inquiry-based, learning process, certain classroom social norms (Cobb & Yackel, 1996) need to be established. Such social norms can include expectations and obligations regarding explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict in interpretations or solutions has become apparent.

In addition to this, certain socio-math norms must be established to create the opportunity for the students to evaluate mathematical progress.

The Design

The goal of the instructional sequence I use as an example is to foster the use of flexible mental computation strategies for addition and subtraction up to 100.4 In designing a conjectured local instruction theory, we can build on the experience gathered in several decades of developmental/design research at the Freudenthal Institute and elsewhere. This research effort has resulted in a domain-specific instruction theory that is grounded in numerous concrete elaborations of the RME approach (Gravemeijer, 1994; Streefland, 1990; Treffers, 1987). By interpreting

---

4Actually, there was a dual goal: linear measurement and flexible arithmetic (e.g., Stephan, Bowers, Cobb, & Gravemeijer, 2004). In this article, however, I limit myself to the arithmetic part.
this domain-specific instruction theory as an instructional design theory, we can point to three design heuristics mentioned previously: guided reinvention (Freudenthal, 1973), didactical phenomenological analysis (Freudenthal, 1983), and emergent modeling (Gravemeijer, 1999).

The design principle of guided reinvention is the key principle of RME. According to Freudenthal (1973), the students should be given the opportunity to experience a process similar to the process by which a given piece of mathematics was invented. For the designer, this implies that a route has to be mapped out that allows the students to invent the intended mathematics by themselves. To do so, the researcher starts with imagining a route by which he or she could have personally arrived at this outcome. In doing so, the designer can take both the history of mathematics and students' informal solution procedures as sources of inspiration.

According to the reinvention principle, the goal of the local instruction theory on addition and subtraction up to 100 is not to teach the students solution strategies in the form of ready-made techniques. Instead, the goal is to help the students develop similar solution methods on their own accounts. A plausible model, then, is to assume that students initially base their computations on their familiarity with certain number relations. When a problem such as \(29 + 5 = \ldots\) has to be solved, number relations involving 29 and 5 can come to mind. A student could think of \(9 + 1 = 10, 29 + 1 = 30, 5 + 4 = 9, 4 + 1 = 5, 5 + 5 = 10\), and so forth and try to use these to solve the problem at hand. One option would be to combine \(29 + 1 = 10\) and \(4 + 1 = 5\) to conclude that you can take 1 of the 5, add that 1 to the 29 to get 30, and add the remaining 4 to the 30 to get 34. Then, from an observer's point of view, it could look like the student is using a building-up-to-10 strategy. For the student, however, this strategy might not be on the horizon yet. Only after reflecting on substantial experience with similar problems, the student could start to notice a pattern and construe the building-up-to-10s strategy. Even then, it could take a while before the student starts to use this strategy as an a priori guidance for choosing a solution procedure.

Thus, the choice of guided reinvention as our point of departure is intertwined with the way we frame our goals. The instructional goal is not to teach the students a set of strategies. Instead, our primary goal is for the students to develop a framework of number relations that offers the building blocks for flexible mental computation.

Having said that, we still can ask ourselves, what solution procedures—or what use of number relations—to aim for. Research on the solution procedures students use to solve addition and subtraction problems up to 100 shows that those procedures fall in two broad categories (Beishuizen, 1993), which we may denote as splitting and counting.

A task like \(44 + 37\), for instance, can be solved in the following manner,

by splitting 10s and 1s:

\[
44 + 37 = \ldots; 40 + 30 = 70; 4 + 7 = 11; 70 + 11 = 81; \text{ or}
\]
by counting in jumps:

\[
\begin{align*}
44 + 37 &= \ldots; 44 + 30 = 74; 74 + 7 = 81; \text{or} \\
44 + 37 &= \ldots; 44 + 6 = 50; 50 + 10 = 60; 60 + 10 = 70; 70 + 10 = 80; 80 + 1 \\
&= 81, \text{or} \\
\text{via some other combination of jumps of 10s and 1s.}
\end{align*}
\]

Beishuizen (1993) found that procedures based on splitting 10s and 1s leads to more errors than solution procedures that are based on counting on and counting back. Moreover, the latter type leaves room for a wide variety of solution procedures and offers more opportunities for curtailing and inventing shortcuts. Counting-by-jumps therefore fits best the type of instructional sequence we aim for.

It can further be noted that, as the example shows, decuples are used as reference points in this counting-by-jumps strategy. In relation to this, decuples also play a central role in framework of number relations that we want the students to develop.

The RME-guided reinvention heuristic is connected with mathematizing; the students invent by mathematizing. The idea is that the students not only mathematize contextual problems—to make them accessible for a mathematical approach—but also mathematize their own mathematical activity, which brings their mathematical activity at a higher level. Freudenthal (1971) characterized mathematizing as a form of organizing, which is also a key element of his didactical phenomenology (Freudenthal, 1983) that constitutes the second design heuristic.

Didactical phenomenology is grounded in a phenomenology of mathematics, within which the focus is on the relation between a mathematical “thought thing” (nooumenon) and the “phenomenon” it describes and analyses, or, in short, organizes.

Phenomenology of a mathematical concept, a mathematical structure, or a mathematical idea means, in my terminology, describing the nooumenon in relation to the phainomena of which it is the means of organizing, indicating which phenomena it is created to organize and to which it can be extended, how it acts upon these phenomena as a means of organizing, and with what power over these phenomena it endows us. (Freudenthal, 1983, p. 28)

In a didactical phenomenology, this relation of mathematical thought thing (concept, structure, or idea) and phenomenon is analyzed from a didactical point of view; the focus is on how the relation is acquired in a learning-teaching process. Freudenthal (1983) contrasted his approach with the (then) conventional approach of trying to concretize abstract concepts (in an “embodiment”). In the latter approach, he concluded, one puts the cart before the horse by teaching abstractions by concretizing them.
What a didactical phenomenology can do is to prepare the converse approach: starting from those phenomena that beg to be organized and, from that starting point, teaching the learner to manipulate these means of organizing. Didactical phenomenology is to be called in to develop plans to realize such an approach. In the didactical phenomenology of length, number, and so on, the phenomena organized by length, number, and so on are displayed as broadly as possible. (Freudenthal, 1983, p. 23)

The didactical phenomenological analysis can orient the researchers toward applied problems that can be suitable as points of impact for a process of progressive mathematization. So, rather than looking around for material that concretizes a given concept, the didactical phenomenology suggests looking for phenomena that might create opportunities for the learner to constitute the mental object that is being mathematized by that very concept.

In relation to a phenomenology of numbers, Freudenthal (1983) noted, “numbers organize the phenomenon of quantity,” whereas “the phenomenon ‘number’ is organized by means of the decimal system” (p. 28). He worked this out in more detail for addition, starting with the lowest level, which is to combine two sets—as in 5 cars and 3 cars or 5 marbles and 3 marbles. However, he argued, problems arise when the addition is not plainly recognizable as the union of two sets, as is the following case: John has 5 marbles, and Pete has 3 more. How many does Pete have? Instead of uniting two given sets, the students must consider the imaginary set of Pete as split into two sets, and reason from there.

Next to those situations, in which addition is not plainly recognizable as the union of two sets, there are also situations in which it is less natural to speak of sets consisting of 5 and 3 elements, such as 5 steps (of stairs) and 3 steps, 5 days and 3 days, or 5 kilometer and 3 kilometer. With those spatial or temporal phenomena one cannot speak of a union of two unstructured sets. Instead, counting is used to organize magnitudes, in which measuring the magnitude is articulated by the natural multiples of a unit. Continuous phenomena are made discrete by a one-to-one mapping of the successive intervals on a sequence of points that follow each other in space or time, in a process that in turn suggests a counting process. In line with this sequential character, the results of additions of magnitudes are obtained by counting on. In relation to this, Freudenthal (1983) pointed to the close relation between cardinal and ordinal numbers: “5 + 3 is defined cardinally, but from olden times it has been calculated ordinarily” (p. 99). The result of 5 + 3 is obtained by starting with the mental 5, and counting on, 6, 7, 8.

From this phenomenological analysis, Freudenthal (1983) concluded, “Counting can and must immediately be transferred from discrete quantities, represented by sets, to magnitudes” (p. 101). He recommended the number line as a device that visualizes magnitudes and, at the same time, the natural numbers. The number line also lends itself to express more or less as directions. In this manner, the number line, or two parallel number lines, can also be used to visualize the problem of Pete
who has three marbles more than John. This reference to the use of the number line brings us to the issue of models and modeling, for example, the next design heuristic.

The third heuristic, the emergent-modeling design heuristic, assigns a role to models that differs from the role of ready-made models as embodiments of abstract concepts mentioned earlier. Instead of trying to concretize abstract mathematical knowledge, the objective is to try to help students model their own informal mathematical activity. The aim is that the model with which the students model their own informal mathematical activity gradually develops into a model for more formal mathematical reasoning. However, the model I am referring to is more an overarching concept than one specific model. In practice, the model in the emergent modeling heuristic is actually shaped as a series of consecutive symbolizations or tools that can be described as a cascade of inscriptions or a chain of signification. From a more global perspective, these tools can be seen as various manifestations of the same model. So when I speak of a shift in the role of the model in the following, I am talking about the model on a more general level. On a more detailed level, this transition can encompass various tools that gradually take on different roles.

The label emergent refers both to the character of the process by which models emerge within RME and to the process by which these models support the emergence of formal mathematical ways of knowing. According to the emergent-models design heuristic, the model first comes to the fore as a model of the students’ situated informal strategies. Then, over time the model gradually takes on a life of its own. The model becomes an entity in its own right and starts to serve as a model for more formal, yet personally meaningful, mathematical reasoning. In relation to this, we discern four different types or levels of activity (Gravemeijer, 1999, 2002):

1. activity in the task setting,
2. referential activity,
3. general activity, and
4. more formal mathematical reasoning.

Although the design of the instructional sequence under discussion is in line with the previously discussed elaboration of a didactical phenomenology of number, it should be noted that Freudenthal’s elaboration was not the actual source for the design. The research team came to similar conclusions on the basis of didactical phenomenological considerations in connection with earlier design experiments. The research team’s didactical-phenomenological deliberations build on the observation that students tend to come up with a wide variety of counting solutions when confronted with linear-type context problems (e.g., Vuurman, 1991). In addition, a closer look at counting strategies shows us that these strategies rely on integrating the cardinal aspect of number (quantity) and the ordinal aspect of number (position/rank). Most addition and subtraction problems concern quantities, whereas the solution procedures consist of moving up and down the number sequence. We argue that it is important that the students connect the first and the latter. This then inspired us to try to integrate measurement and the empty number line.

I use the word “tools” as a generic term in the following discussion, encompassing also symbolizations, or inscriptions.
These levels of activity underline that the model is grounded in students' understandings of paradigmatic, experientially real task settings. In other words, the model emerges as situation-specific imagery. This implies that initially, at the referential level, the model is meaningful for the students because it signifies for them the activity in the task setting to which it refers. General activity begins to emerge as the students start to reason about the mathematical relations involved. As a consequence, the model loses its dependency on situation-specific imagery and gradually develops into a model that derives its meaning from a framework of mathematical relations that is being construed in the process. The transition from model-of to model-for coincides with a progression from informal to more formal mathematical reasoning that involves the creation of new mathematical reality, which is thought of as consisting of mathematical objects (Sfard, 1991) within a framework of mathematical relations. The level of more formal activity is reached when the students no longer need the support of models.

As an aside, it can be noted that this transition cannot be pinned down to one specific symbolization or tool. Instead, there is a gradual change in the way the students perceive and use tools as their personal framework grows.

Several authors have proposed the use of the number line as a means of support for addition and subtraction up to 100 (Freudenthal, 1983; Treffers & de Moor, 1990; Whitney, 1988). For me, the objective to help students make a connection between the cardinal and the ordinal aspect forms the main argument to introduce the number line as a tool. From an expert's point of view, a number line integrates both the cardinal aspect (line segment) and the ordinal aspect (point). In addition to this, the number line offers a way of symbolizing that fits nicely the various counting strategies—by describing the subsequent counting steps as arcs on an empty number line (Gravemeijer, 1994, 1999). I speak of an empty number line because this number line is empty except for the numbers that are actually needed. The students add these to the number line as a part of the solution process (see Fig. 3).

The interpretation of the number line, however, is not self-evident. For the students, it does not speak for itself what the marks on the number line signify. The hash marks might signify either cardinal or ordinal numbers and not necessarily a synthesis of the two. Exactly for that reason, Whitney (1988) and Treffers (Treffers & de Moor, 1990) let their introduction of the number line be preceded by activities with a

![FIGURE 3 Solving 38 + 24 on the empty number line.](image-url)
bead string. This bead string consists of 100 beads, colored in groups of 10. While executing various counting tasks, students find out that the decimal structure can be used to solve tasks like, “Count 38 and add 24 more. Which number do you get?” This solution procedure is being modeled with arcs on the empty number line. The indispensability of this kind of imagery proved itself in a design experiment where the bead string was skipped (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). The students did not have the means to resolve among themselves whether a hash mark with a number, say 38 for example, should be thought of as signifying 38 objects (e.g. candies or beads) or the thirty-eighth object.

I concur with Freudenthal’s (1983) arguments to ground addition and subtraction in linear measurement, as measuring presents itself as a natural alternative for counting on a bead string. Conceptual understanding of measurement requires that students interpret the activity of measuring as the accumulation of distance (Thompson & Thompson, 1996). Similarly, a number on a ruler would have to signify the total measure of the distance measured. Speculating on the genesis of the ruler in history, one can take the view that the ruler came about as a curtailment of iterating a measurement unit. So, the ruler can be thought of as a model of iterating some measurement unit, whereas the empty number line can function as a model for more sophisticated mathematical reasoning in the context of mental computation strategies with numbers up to 100. The connection between the two can be made by building on the relation between iterating measurement units as accumulating distances and a cardinal interpretation of positions on the number line. This is truly a model-of/model-for transition if it coincides with a shift in the student’s view of numbers as referents of distances to numbers as mathematical entities. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i.e., numbers as constituents of magnitudes, such as 38 feet) to viewing numbers as mathematical objects (e.g., 38). For the student, a number viewed as a mathematical object still has quantitative meaning, but this meaning is no longer dependent upon its connection with identifiable distances or with specified countable items. Instead, numbers viewed as mathematical objects derive their meaning from their place in a network of number relations.

The enacted instructional sequence. With the help of the previously discussed elaboration of the design heuristics, we developed a preliminary design of the instructional sequence, which was worked out in the teaching experiment in Nashville, Tennessee, which is well-documented in various publications (e.g., Gravemeijer, 1999; Stephan, 1998; Stephan et al., in press; Stephan, Cobb, Gravemeijer, & Estes, 2001). Space does not allow for an elaborate account of all findings here. Instead, I give a brief description of the enacted instructional sequence, supplemented with elements of the retrospective analysis that offer essential background theory for teachers. With the latter, I want to highlight the importance of offering teachers more than a set of instructional activities. Note that I
present a somewhat smoothed description of the enacted sequence in which encountered detours are left out.

The sequence starts with a story of a country where the king's foot serves as the measurement unit and the king does all the measuring. The students follow the king's example by pacing various items in the classroom (heel-to-toe). In the sequel to this story, an alternative is sought to the king doing all the measuring. After some discussion a footstrip of five feet is introduced and the students start measuring with this footstrip. Later, a new scenario is introduced, which is about Smurfs (small blue dwarfs) who measure with food cans. Those food cans happen to have the same size as the Unifix cubes that are available in the classroom. The students measure objects by stacking Unifix cubes until they reach the required length. As the story unfolds, measuring with individual cubes is exchanged for measuring with a "Smurfbar," consisting of 10 Unifix cubes, which implies coordinating 10s and 1s. In the context of the story, this Smurfbar is invented to free the Smurfs from the task of carrying numerous food cans with them. To get rid of the obligation to carry food cans completely, a paper "10-strip" is made to replace the Smurfbar. Finally, ten 10-strips are pasted together to construe a measurement strip of 100 cubes long. Next, a significant step is taken, when the activity of measuring with the measurement strip is followed by tasks about incrementing, decrementing, and comparing lengths. Here the students have to take a length as a given and use the measurement strip as a means of support for solving the problems. Gradually, the counting strategies that the students use to solve those problems are replaced by forms of arithmetical reasoning that build on the measurement strip. Instead of counting the difference between 34 and 56, for instance, the students may reason $34 + 6 = 40$, $40 + 10 = 50$, and $50 + 6 = 56$, so the difference is $6 + 10 + 6 = 22$. When the students have reached this stage, the empty number line is introduced as a means of scaffolding and communicating the arithmetical solution procedures that the students use. As a last step, the empty number line notation is generalized to support arithmetical reasoning in contexts other than measuring.

The local instruction theory. The previously discussed design experiment resulted in the local instruction theory that is summarized in Table 1, which gives an overview of the potential tool use, the corresponding imagery and mathematical activity, and topics of mathematical discourse. I do not lay out this instruction theory in detail, nor do I underpin it with an account of the retrospective analysis of the learning process of the classroom community. All this is described in detail in Stephan et al. (in press). Instead, I use elements of the retrospective analysis as a basis for an elaboration of two more fundamental dimensions of the researchers' theory on how the instructional activities can work, which teachers must come to grips with to be able to design hypothetical learning trajectories for their own

---

2Because Unifix cubes can be clicked together, it is rather easy to make a solid stack of cubes.
<table>
<thead>
<tr>
<th>Tool</th>
<th>Imagery</th>
<th>Activity/T-a-s Interests</th>
<th>Potential Mathematical Discourse Topics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feet (heel to toe)</td>
<td>Measuring</td>
<td>Reasoning about activity of pacing</td>
<td>Focus on covering distance</td>
</tr>
<tr>
<td>Masking tape</td>
<td>Record of activity of pacing</td>
<td>Reasoning about activity of pacing</td>
<td>Measuring as divorced from activity of measuring</td>
</tr>
<tr>
<td>Footstrip</td>
<td>Record of pacing (builds on masking tape) (Form/function shift: using a record of pacing as a tool for measuring)</td>
<td>Measuring with a “big step” of five = measuring by iterating a collection of paces</td>
<td>Structuring distance in collections of 5s and 1s</td>
</tr>
<tr>
<td>Smurf cans</td>
<td>Stack of Unifix cubes signifies result of iterating</td>
<td>Measuring by creating a stack of Unifix cubes</td>
<td>Builds on measuring divorced from activity of iterating</td>
</tr>
<tr>
<td>Smurf bar</td>
<td>Signifies result of iterating</td>
<td>Measuring by iterating a collection of 10 Unifix cubes</td>
<td>Accumulation of distances</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Structuring distance into measures of 10s and 1s</td>
<td>Coordinating measuring with 10s with measuring by 1s</td>
</tr>
<tr>
<td>10-strip</td>
<td>Signifies measuring 10s and 1s with the Smurf bar</td>
<td>Measuring by iterating the 10-strip, and using the strip as a ruler for the 1s</td>
<td>Accumulation of distances</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Coordinating 10s &amp; 1s</td>
</tr>
<tr>
<td>Measurement strip</td>
<td>Signifies measuring with 10 strip/Starts to signify result of measuring (Form/function shift: inscription developed for measuring is used for scaffolding and communicating)</td>
<td>(1) Measuring: strip alongside item; counting by 10s and 1s</td>
<td>Distance seen as already partitioned; extension already has a measure</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2) Reasoning about spatial extensions (results of measuring have become entities in and of themselves)</td>
<td>Part-whole reasoning/quantifying the gaps between two or more lengths</td>
</tr>
<tr>
<td>Empty number line</td>
<td>Signifies reasoning with measurement strip</td>
<td>Means of scaffolding &amp; means of communicating about reasoning about number relations</td>
<td>Numbers as mathematical entities (numbers derive their meaning from a framework of number relations)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Various arithmetical strategies</td>
</tr>
</tbody>
</table>

*Note.* Reprinted with permission from Stephan, M., Bowers, J., Cobb, P., & Gravemeijer, K. (Eds.), *Supporting students’ development of measuring conceptions: Analyzing students’ learning in social context. Journal for Research in Mathematics Education Monograph No. 12.* Copyright 2003 by the National Council of Teachers of Mathematics. All rights reserved.
classrooms. The first concerns an empirically grounded theory on how the students are expected to make sense of acting with new tools and how this relates to preceding activities. The second concerns an empirically grounded theory on the students’ conceptual development in relation to the relevant mathematical concepts.

Tools and Imagery
In RME design, the emergent model and the corresponding series of tools function as the backbone of the intended reinvention process. Ideally, the students should invent the necessary tools for themselves. This, however, is not really feasible. We take care, however, that the students are involved in the invention process. This can be done by a careful introduction of each new tool according to the following set up: Each new tool has to come to the fore as a solution to a problem (e.g., how to measure more efficiently). First, the students are given the opportunity to think about and discuss possible solutions to that problem, then the students are asked to evaluate whether the new tool offers an acceptable solution to the problem. In this manner, the students experience an involvement in the invention process even though they do not invent the tools for themselves. In this manner, we try to ensure that the tools emerge in a sense from the activity of the students. In addition, we make sure that the use of new tools is grounded in some imagery for the students. That is to say, there has to be some history in the learning process of the student that renders meaning to the activity with a new tool. I already pointed to this issue of imagery in relation to the interpretation of the hash marks on the number line. The brief sketch of the enacted instructional sequence reveals the following series of tools that the students use: feet, footstrip, Smurf cans, Smurf bar, 10-strip, measurement strip, and empty number line.

The first link concerns pacing with one’s feet and measuring with the footstrip. In practice, the connection between the two was mediated by the construction of a record of the activity of pacing. The teacher made this record to facilitate the discussion of the different ways the students were counting the placement of their feet, when measuring by pacing heel-to-toe. The teacher placed pieces of masking tape at the beginning and end of each pace, which enabled her—and the students—to point to the various paces after the fact. With help of this record, the students discussed whether the first foot should be counted. Thanks to this history, this record signified the activity of pacing for the students. The footstrip that is introduced later builds on this notion of a record of pacing as the footstrip, too, can be seen as a record of pacing. But then a form/function shift (Saxe, 1991) takes place when the students start using this record of pacing as a tool for measuring. However, as measuring with the footstrip builds on the imagery of the activity of pacing with individual feet, putting down the footstrip can signify pacing heel-to-toe for the students.

When the students start measuring with the Unifix cubes in the Smurf scenario, there is no direct link to earlier activities in terms of imagery. The learning history of the students, however, does play a role because the experience of pacing and
measuring with the footstrip enables the students to consider a stack of Unifix cubes as a result of measuring, an observation that was underscored by the students’ inability to do this when we tried to introduce the Unifix cubes too quickly.

The transition from measuring with individual cubes to measuring with the Smurf bar is similar to the transition between pacing and measuring with the footstrip. Next, measuring with the 10-strip builds on the history of measuring with the Smurf bar. Thanks to this history, measuring with the 10-strip signifies measuring 10s and 1s with the Smurf bar for the students. In turn, measuring with the measurement strip builds on the imagery of measuring with the 10-strip. Actually, the students initially count by 10s and 1s on the measurement strip to establish the length of an item. Gradually, however, a position on the measurement strip starts to signify the result of measuring. Then, we meet another form/function shift, when the tasks shift from measuring to reasoning about measures, and the measurement strip is used as a means for scaffolding and communicating ways of reasoning. Finally, drawing arcs on the empty number line is introduced as an alternative means for scaffolding and communicating ways of reasoning with the measurement strip.

The main idea behind the design of this cascade of tools is that the way in which the students act and reason with each tool builds on their activity with earlier ones. This build up is to ensure that the students have a meaningful way of acting with the tools, because they can rely on the imagery of acting with earlier, already familiar tools. From this perspective, it is essential that the teachers who want to reenact the sequence come to grips with the researchers’ empirically grounded theory on how reasoning with one tool builds upon the other.

Potential Mathematical Discourse Topics

A significant feature of the instruction, described previously, is that the agenda of the designers differs from the immediate goals of the tasks. The immediate goal for the students is, for instance, to figure out how long something is or what the difference between two lengths is. But the instructional objective is to create a situation that gives rise to various solution strategies, which in turn lend themselves to a discussion on significant mathematical issues. It is therefore important that teachers, who intend to reenact this sequence, understand these potential mathematical discourse topics and their relation to the intended mathematical development of the students. The potential mathematical discourse topics listed in Table 1 are taken from the doctoral thesis of Michelle Stephan (1998), which discerned a series of mathematical practices that reflect the way in which taken-as-shared ways of reasoning, arguing, and tool use evolved as the sequence was enacted. Those mathematical practices both encompass the way of acting and reasoning with tools and the conceptual understanding involved.

The instructional sequence starts with students measuring objects by pacing. However, for some of the students, the goal appeared to be just to count the number of
steps it took them to reach the end of the item. This was inferred from the fact that some students did not count the placement of the first foot (when the heel was aligned with the beginning of the item measured). They started counting, "one," with the placement of the second foot, whereas others started their counting with the first foot. What we are aiming for is that the students come to see measuring as covering amounts of space. To reach this goal, the teacher has to make the two different ways of measuring a topic of discussion (whether you have to count the first foot). In such a discussion, the students can start to realize that it is not just a matter of convention; instead, if one does not count the first foot, an amount of the item would not be measured. In this manner, students can come to see the goal of measuring as covering amounts of space—as was the case in the experimental classroom.

With the activity of measuring with a footstrip of five paces, many students ran into problems when the space that was being covered by the footstrip extended past the physical extension of the measured item. For them, apparently, measuring was tied to the physical act of placing a footstrip, and they could not mentally cut the footstrip when needed. The instructional goal here is that the space to be measured takes priority over the measurement activity and becomes independent of activity for the students. Whole-class discussions on concrete instances are needed to create opportunities for students to articulate how the extended footstrip can be mentally cut.

The activity of measuring with a Smurf bar in the Smurf scenario showed that, for some students, the curtailment of counting by individual cubes was based only on a number word relation. For instance, when the second iteration of the Smurf bar would extend beyond the item measured, they would count the cubes past the first iteration as "21, 22, 23 and so forth", instead of, "11, 12, 23, ..." For them, "20" seemed to be the number word associated with the second placement of the Smurf bar rather than the amount of space covered by 20 cubes. Although what is aimed for is that the students realize that as "20" signifies the length covered by 20 cubes, 21, 22, and 23 must extend beyond the length whose measure was 20. In other words, the students have to come to grips with coordinating measuring with 10s with measuring with 1s. To achieve this, the latter has to become a topic of discussion in the classroom.

In subsequent activities, the Smurf bar is replaced by paper 10-strips and next ten 10-strips are taped together to make a measurement strip of 100 cans long. Here, the students may initially measure with the measurement strip by laying the strip down alongside the item and counting by 10s and 1s until they reach the endpoint of the item. Gradually, however, the students curtail their activity of counting up on the measurement strip and find that the length of an item can be measured by laying down the measurement strip alongside the item and simply reading off the numeral corresponding to the position of the farthest endpoint. To do this insightfully, they have to conceive an extension as already having a measure, independent of the activity of measuring.

The next set of activities involves tasks such as comparing the lengths of two items and figuring out the difference with the help of the measurement strip. This is
the first instructional activity in which the students do not measure an item that is physically present. The mathematically significant issue here is the quantification of a gap between two numbers. In the teaching experiment, some students counted the spaces between the two numbers whereas others counted the lines. To overcome this problem, the teacher made the different ways of quantifying the gaps a topic of discussion and asked the students to explain what each line or space signified to them. As a result of such discussions, the method of counting spaces to specify the measure of the spatial extension between two lengths became taken-as-shared.

A next step that is aimed for is that the students gradually replace the method of literally counting spaces by arithmetical reasoning. Again the teacher plays an important role by stimulating discussions on the different ways of establishing the number of spaces.

Finally, the empty number line is introduced as a means of describing and scaffolding various forms of arithmetical reasoning. When making the transition from the measurement strip to the number line, it is essential that the students differentiate between the activity of measuring and the activity of representing arithmetical strategies. On the empty number line the goal is for students to express how they would, for instance, increment 64 with 28. For example, by first measuring 64, then adding six 1s, which would get one to 70, then measuring two times 10, which would result in 80 and 90, respectively, and finally adding two 1s, which adds up to 92. When describing this method, it would be sufficient to show that when starting at 64, add 6, arrive at 70; then add 10, arriving at 80; another 10, arriving at 90; and 2, arriving at 92. To strive for an exact proportional representation of all the jumps would severely hamper a flexible use of the number line. Therefore we must make sure that the students are aware of the distinction between the ruler as a measurement tool and the empty number line as a means of describing solution procedures. Thus when they make drawings, the intention of the students should not be to make a schematic drawing of a measuring device, but to make a drawing that shows their arithmetical reasoning.

What is expected is, that in the course of the sequence, a shift is taking place in which the student’s view of numbers transitions from referents of distances to numbers as mathematical entities. As argued before, this shift involves a transition from viewing numbers as tied to identifiable objects or units to viewing numbers as entities on their own that derive their meaning from a framework of number relations. This framework of number relations, then, offers the basis for flexible mental computation strategies for addition and subtraction up to 100, which was our instructional goal.

**CONCLUSION**

The main issue of this article is what instructional design has to offer to reform mathematics education, whereas classical instructional design theories do not fit
mathematics education that tries to capitalize on the inventions of the students. The classical approach of task analysis results in a breakdown of the mathematical content in a hierarchy of small learning objectives that have to be mastered in a fixed sequence. This sequence is to be followed independent of the input or interest of the students; the only variation is one in speed and reteaching. A final drawback of the analytically defined learning objectives is that the students cannot see the relevance until they have reached the end of the process.

Still, I argue, if justice is to be given to the input of the students and their ideas built on, a well-founded plan is needed. In this respect, I point to the proactive role of the teacher in establishing an appropriate classroom culture, in choosing and introducing instructional tasks, organizing group work, framing topics for discussion, and orchestrating discussion. Following Simon (1995), this implies designing, enacting, assessing, and revising hypothetical learning trajectories in an iterative series of mathematical teaching cycles.

I use the example of the local instruction theory on addition and subtraction to show that design research can help teachers by developing viable local instruction theories, which can be used by classroom teachers to construe hypothetical learning trajectories that fit the characteristics and actual situations of their own classrooms. I highlight the word theory because, in contrast with traditional design theories, the emphasis is not on an elaborated instructional sequence with detailed directions for the teacher, but on the theory that underpins a possible instructional sequence—a theory of which we claim offers an empirically grounded theory on how the instructional sequence can work. The examples of the theory behind the way the various tools build on each other and the theory on how the conceptual development of the students can be supported by exploiting potential mathematical topics for discussion shed light on the theoretical framework that teachers need to make informed decisions in the classroom. In line with the RME theory that inspired the design, this enables teachers to design instruction that helps students to develop their current ways of reasoning into more sophisticated ways of mathematical reasoning.

REFERENCES


